MTH 201
Multivariable calculus and differential equations
Homework 8
Surface area, vector fields, line integral

Surface area

1. Find the surface area of
   (a) $z = 1 + 2x + 3y$ where $1 \leq x \leq 4$ and $0 \leq y \leq 1$.
   (b) the part of the surface $z = x^2 + 2y$ that lies above the triangular region in the $xy-$plane with vertices $(0,0), (1,0)$, and $(1,1)$.
   (c) the part of the plane $z = 6 - 3x - 2y$ that lies in the first octant.
   (d) the part of the surface $z = xy$ that lies in the cylinder given by $x^2 + y^2 = 1$.
   (e) the part of the surface $z = x^2 + y^2$ that lies under the plane $z = 9$. [HW]

2. Find a parametric representation for
   (a) the cylinder $x^2 + y^2 = 4, 0 \leq z \leq 1$.
   (b) the elliptic paraboloid $z = x^2 + 2y^2$.

3. Describe the surfaces given by the following:
   (a) $\overrightarrow{r}(\theta, t) = (a\cos\theta, a\sin\theta, t)$ where $0 \leq \theta \leq 2\pi, -1 \leq t \leq 1$, and $a > 0$ is fixed.
   (b) $\overrightarrow{r}(\rho, \theta) = (\rho\cos\theta\sin\alpha, \rho\sin\theta\sin\alpha, \rho\cos\alpha)$ where $0 \leq \rho \leq 1, 0 \leq \theta \leq 2\pi$, and $\alpha = \frac{\pi}{4}$ is fixed. [HW]
   (c) $\overrightarrow{r}(\theta, \phi) = (\rho\cos\theta\sin\phi, \rho\sin\theta\sin\phi, \rho\cos\phi)$ where $0 \leq \theta \leq 2\pi, 0 \leq \phi \leq \pi$, and $\rho > 0$ is fixed.

4. Find the surface area of each of the surfaces described in the previous question.

5. Find the surface area of the unit hemisphere defined by $z = \sqrt{1 - x^2 - y^2}$, where (x, y) lies in the circular region $T = \{(x, y) : x^2 + y^2 \leq 1\}$. [HW]

Vector fields and line integral

6. Sketch the vector field $\mathbf{F}$ given by
   (a) $\mathbf{F}(x, y) = \frac{x}{2}\overrightarrow{i} + y\overrightarrow{j}$
   (b) $\mathbf{F}(x, y) = \frac{y}{\sqrt{x^2+y^2}}\overrightarrow{i} + \frac{x}{\sqrt{x^2+y^2}}\overrightarrow{j}$
   (c) $\mathbf{F}(x, y) = \frac{y}{\sqrt{x^2+y^2}}\overrightarrow{i} - \frac{x}{\sqrt{x^2+y^2}}\overrightarrow{j}$
   (d) $\mathbf{F}(x, y, z) = x\overrightarrow{i}$
   (e) $\mathbf{F}(x, y, z) = 2x\overrightarrow{i} - 2y\overrightarrow{j} - 2x\overrightarrow{k}$

7. Find the gradient vector field of the following functions
   (a) $f(x, y) = x^2 + y^2$
   (b) $f(x, y) = x^2 \sin 5y$

(c) $f(x, y, z) = ze^{-xy}$.

8. Evaluate the line integral $\int_C f ds$, where
   (a) $f(x, y) = x^2 - y^2$ and C is the part of the circle $x^2 + y^2 = 4$ from $(2, 0)$ to $(\sqrt{2}, \sqrt{2})$ in the first quadrant. [HW]
   (b) $f(x, y, z) = \sqrt{x^2 + y^2}$ and C is the curve given by $\overrightarrow{r}(t) = 4\cos t\,\overrightarrow{i} + 4\sin t\,\overrightarrow{j} + 3t\,\overrightarrow{k}, 0 \leq t \leq \pi/2$.
   (c) $f(x, y, z) = 3x^2 - 2y + z$ and C consists of a line segment from $(0, 0, 0)$ to $(2, 2, 0)$ followed by a vertical line segment from $(2, 2, 0)$ to $(2, 2, 2)$.

9. Compute $\int_C xydx + yzdy + zxdz$, where C is the smooth path given by $\overrightarrow{r}(t) = t\,\overrightarrow{i} + t^2\,\overrightarrow{j} + t^3\,\overrightarrow{k}, 0 \leq t \leq 1$.

10. Compute the line integral $\int_C \mathbf{F} \cdot dr$, where
   (a) $\mathbf{F}(x, y) = \frac{x}{\sqrt{x^2+y^2}}\overrightarrow{i} + \frac{y}{\sqrt{x^2+y^2}}\overrightarrow{j}$ and C is the parabola $y = 1 + x^2$ from $(-1, 2)$ to $(1, 2)$.
   (b) $\mathbf{F}(x, y, z) = (y + z)\overrightarrow{i} + (z + x)\overrightarrow{j} + (z + x)\overrightarrow{k}$ and C consists of a line segment from $(0, 0, 0)$ to $(1, 0, 1)$ followed by a line segment from $(1, 0, 1)$ to $(0, 1, 2)$.
   (c) $\mathbf{F}(x, y, z) = (y \sin z)\overrightarrow{i} + (z \sin x)\overrightarrow{j} + (x \sin y)\overrightarrow{k}$ and C is given by $\overrightarrow{r}(t) = \cos t\,\overrightarrow{i} + \sin t\,\overrightarrow{j} + \sin 5t\,\overrightarrow{k}, 0 \leq t \leq 2\pi$. [HW]

11. Compute the work done by the force field $\mathbf{F}(x, y) = x^2\overrightarrow{i} - xy\,\overrightarrow{j}$ in moving an object along the quarter circle $\overrightarrow{r}(t) = \cos t\,\overrightarrow{i} + \sin t\,\overrightarrow{j}, 0 \leq t \leq \frac{\pi}{2}$. [HW]